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D-brane charges on non-simply connected groups

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Abstract

The maximally symmetric D-branes of string theory on the non-simply connected Lie group $\mathrm{SU}(n)/\mathbb{Z}_d$ are analysed using conformal field theory methods, and their charges are determined. Unlike the well understood case for simply connected groups, the charge equations do not determine the charges uniquely, and the charge group associated to these D-branes is therefore in general not cyclic. The precise structure of the charge group depends on some number theoretic properties of n, d, and the level of the underlying affine algebra k. The examples of $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ and $\mathrm{SU}(3)/\mathbb{Z}_3$ are worked out in detail, and the charge groups for $\mathrm{SU}(n)/\mathbb{Z}_d$ at most levels k are determined explicitly.

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1. Introduction

It is believed that the topological charges of D-branes are described by some K-theory group [1,2]. These charges constrain the dynamics of D-branes as configurations that carry different charges cannot decay into one another. For the case of the WZW models, the relevant K-groups are believed to be certain twisted K-theory groups [3,4].

The D-brane charge groups are also calculable in terms of a microscopic (conformal field theory) description of D-branes. Let us denote by $\mathcal{B}$ the (finite) set of D-branes that preserve the full affine symmetry $\mathfrak{g}_k$ (possibly up to an automorphism). The charge $q_a \in \mathbb{Z}$ of the D-brane $a \in \mathcal{B}$ must then satisfy the charge equation [5]

$$\dim(\lambda)\, q_a = \sum_{b\in\mathcal{B}} \mathcal{N}_{\lambda a}{}^{b}\, q_b\,. \tag{1.1}$$

Here $\lambda \in P_+^k(\mathfrak{g})$ is an integrable highest weight of $\mathfrak{g}_k$, $\dim(\lambda)$ is the Weyl dimension of the corresponding finite dimensional representation of the horizontal subalgebra $\bar{\mathfrak{g}}$, and $\mathcal{N}_{\lambda a}{}^{b}$ are the NIM-rep coefficients that describe the multiplicity (possibly zero) with which the representation λ appears in the open string spectrum of an open string that begins on the D-brane a, and ends on the D-brane b.

In the simplest situation (namely for the D-branes in the charge conjugation modular invariant that preserve the full affine algebra without any automorphism) the D-branes are labelled by the integrable highest weight representations, $a=\mu$, and the NIM-rep $\mathcal{N}_{\lambda\mu}{}^{\nu}$ agrees with the fusion rules $N_{\lambda\mu}{}^{\nu}$. The charges q_μ are then (up to trivial rescalings) uniquely determined by (1.1), $q_\mu = \dim(\mu)$, and they satisfy the charge equation modulo an integer M

$$\dim(\lambda)\dim(\mu) = \sum_{\nu\in P_+^k} N_{\lambda\mu}{}^{\nu}\dim(\nu) \qquad (\mathrm{mod}\ M)\,. \tag{1.2}$$

The integer M has been determined for all algebras and levels [5,6,7], and it is given by the universal formula

$$M = \frac{k+h^\vee}{\gcd(k+h^\vee, L)}\,, \tag{1.3}$$

where $h^\vee$ is the dual Coxeter number of the finite dimensional Lie algebra $\bar{\mathfrak{g}}$, and L depends only on $\bar{\mathfrak{g}}$. For the case of su(n) that shall mainly concern us in this paper,

$$M = M_{\mathrm{SU}(n)} = \frac{n+k}{\gcd(n+k, L)}\,, \tag{1.4}$$

where $L = \mathrm{lcm}\{1,2,\ldots,n-1\}$.

Since the coefficients $\mathcal{N}_{\lambda a}{}^{b}$ form a (NIM-)representation of the fusion rules, any solution to (1.1) can only be satisfied mod M, as is explained in [8]. In general, however, the solutions of (1.1) will not be unique, and the set of all solutions to (1.1) forms a $\mathbb{Z}_M$-module K. [This is to say, the sum of two solutions is a solution, and replacing q_a by lq_a (where $l \in \mathbb{Z}_M$) also leads to a solution.] Since there are only finitely many D-branes (and since M is a finite number), K will only contain finitely many elements.

The set of solutions K is always an abelian group (under addition): the identity element is the trivial charge solution ($q_a = 0$ for all a), and the inverse to the solution q_a is the solution $-q_a$. This group is the *charge group* of the set of D-branes whose open string spectrum is described by the NIM-rep $\mathcal{N}_{\lambda a}{}^{b}$. Any finite abelian group is of the form

$$K = \mathbb{Z}_{M_1} \oplus \mathbb{Z}_{M_2} \oplus \cdots \oplus \mathbb{Z}_{M_l} \, . \tag{1.5}$$

Apart from the trivial identification $\mathbb{Z}_a \oplus \mathbb{Z}_b = \mathbb{Z}_{ab}$ whenever a and b are coprime, this decomposition is unique. Since K is a $\mathbb{Z}_M$-module, all M_i must be factors of M. The charge group of the D-branes is therefore always of this form.

Strings propagating on the simply connected group manifold G are described, in terms of conformal field theory, by the charge conjugation modular invariant. For this theory, the charge group for the D-branes that preserve the full affine symmetry has been determined [5,6,7], and it is just $K = \mathbb{Z}_M$, with M given by (1.3) above. For the D-branes that only preserve the affine symmetry up to an outer automorphism, the charge group has recently also been found to be $K = \mathbb{Z}_M$ [8]. These results are in beautiful agreement with the K-theory calculations of [9,10].

In this paper we want to determine the D-brane charge group for the case of string theory on a group manifold that is not simply connected. We shall only consider the case of quotient groups of the simply connected group SU(n), although many of our statements generalise directly to the other cases. Recall that the centre of SU(n) is $\mathbb{Z}_n$. To each divisor $d > 1$ of n, there is therefore an associated compact connected (but not simply connected) Lie group $G = \mathrm{SU}(n)/\mathbb{Z}_d$. Moreover, any compact connected Lie group with the same Lie algebra su(n) as SU(n) will be of this form. The modular invariant corresponding to G is known [11,12] and will be reviewed in the following subsection.

We shall only consider the 'untwisted' D-branes in this paper, *i.e.* the branes that preserve the full affine symmetry without any (outer) automorphisms. These have been implicitly constructed in [13] (see also [14,15] for earlier work), but we will need to be more

explicit here, and shall review the relevant details below as well. As we shall see, unlike the situation in the simply connected case, the charges here will not be uniquely determined by (1.1), and K is therefore not a cyclic group in general.

As will become apparent in the subsequent analysis, there are two classes of solutions to (1.1), which together generate all solutions. One solution (that we shall call 'untwisted') is characterised by the property that $q_0 = 1$, where $a = 0$ describes the D0-brane of the underlying $\mathrm{SU}(n)$ theory (that also defines a D-brane of $\mathrm{SU}(n)/\mathbb{Z}_d$). Provided that $n(n+1)/d$ is even, this leads to a summand of $\mathbb{Z}_{M_{\mathrm{SU}(n)}}$ in (1.5). [This is for example illustrated by the example of $\mathrm{SU}(3)/\mathbb{Z}_3$ that will be discussed very explicitly in section 2.] This charge solution simply measures the (rescaled) D0-brane charge of the underlying $\mathrm{SU}(n)$ theory.

If $n(n+1)/d$ is odd on the other hand, the situation is different in that the D0-brane of $\mathrm{SU}(n)/\mathbb{Z}_d$ (and therefore almost all D-branes of $\mathrm{SU}(n)/\mathbb{Z}_d$) do not carry *any* non-trivial D0-brane charge with respect to the underlying $\mathrm{SU}(n)$ theory. In this case a non-trivial solution with $q_0 = 1$ may still exist (although this need not be the case), but it does not correspond to the D0-brane charge of the underlying $\mathrm{SU}(n)$ theory any more. In particular, it only gives rise to a summand of $\mathbb{Z}_{M^u}$ in (1.5), where M^u is typically either $M^u = 1$ or $M^u = 2$, and at any rate does not grow with k any more. This 'pathological' behaviour already occurs for the simplest non-simply connected Lie group $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$; this example will be discussed very explicitly in section 2.3.

In many cases the above untwisted solution is the unique solution to (1.1) (in particular, this is the case provided that the level of the affine Lie algebra k is coprime to d), but there are also situations where $q_0 = 1$ does not determine the full solution uniquely. The ambiguity can then be described by 'twisted' solutions for which $q_0^t = 0$. For example, if $d > 2$ is prime (and k is a multiple of d), then there are precisely $d - 1$ (linearly independent) twisted solutions, each of which contributes a factor of

$$\mathbb{Z}_{M^t} \qquad \text{where} \qquad M^t = \gcd(d^\infty, n, M_{\mathrm{SU}(n)}) \tag{1.6}$$

to (1.5). If d is composite (or $d = 2$), the precise structure of the twisted solutions is more complicated, but the general structure is similar. The structure of the twisted solutions is nicely illustrated by the example of $\mathrm{SU}(3)/\mathbb{Z}_3$.

We are not able to give a proof for these statements in all cases, but we can give a nearly complete description when d is a prime, and when d is composite but coprime to

$M_{\mathrm{SU}(n)}$. We have also studied a number of examples in detail, and the outlines of the general picture are taking shape. One would expect that these charge groups should agree with the appropriate twisted K-theory groups; it would be very interesting to confirm this by a direct K-theory analysis.

The paper is organised as follows. In the remainder of this section we introduce our notation and review the construction of the relevant conformal field theories, as well as their untwisted D-branes. In section 2 we illustrate our results by working out the examples of $\mathrm{SU}(3)/\mathbb{Z}_3$ and $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ explicitly. In section 3 we collect some general observations and explain for which theories the charge group is necessarily pathologically small. We also show how to construct the unique (untwisted) solution provided that k is coprime to d. Finally we prove that the untwisted solution always leads to $\mathbb{Z}_{M_{\mathrm{SU}(n)}}$ provided that d is coprime to $M_{\mathrm{SU}(n)}$ (and $n(n+1)/d$ is even so that the solution is not pathologically small). In section 4, we give a nearly complete description of the charge group for the case when d is prime and comment on the generalisations to composite d. Finally, section 5 contains our conclusions and conjectures about general $\mathrm{SU}(n)/\mathbb{Z}_d$.

1.1. A quick review of SU(n) and its simple current modular invariants

String theory on $\mathrm{SU}(n)$ can be described in terms of the representation theory of the affine algebra $\widehat{A}_{n-1} = \widehat{\mathrm{su}}(n)$ at the appropriate level k. Its integrable highest weights $\lambda \in P_+^k(\mathrm{su}(n))$ consists of all n-tuples $(\lambda_0; \lambda_1, \ldots, \lambda_{n-1})$, where each λ_i is a nonnegative integer, and $\sum_{i=0}^{n-1} \lambda_i = k$. When there can be no confusion about the level k, we will often drop the redundant component λ_0. These weights $\lambda \in P_+^k(\mathrm{su}(n))$ parametrise the primary fields of the WZW model on $\mathrm{SU}(n)$.

This WZW model has a simple current J of order n, corresponding to the cyclic symmetry of the extended Dynkin diagram, which sends the highest weight $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_{n-1})$ to $J\lambda = (\lambda_{n-1}; \lambda_0, \lambda_1, \ldots, \lambda_{n-2})$. This permutation obeys [16,17]

$$S_{\lambda\, J^j \mu} = e^{2\pi \mathrm{i} j\, t(\lambda)/n}\, S_{\lambda\mu} \tag{1.7}$$

for any j, where $t(\lambda)$ is its *n-ality*

$$t(\lambda) = \sum_{j=1}^{n-1} j \lambda_j \,. \tag{1.8}$$

Simple currents give rise to symmetries and gradings of fusion coefficients

$$\begin{aligned} N_{J^i\lambda, J^j\mu}{}^{J^{i+j}\nu} &= N_{\lambda\mu}{}^{\nu} \\ N_{\lambda\mu}{}^{\nu} \neq 0 \implies t(\lambda) &+ t(\mu) = t(\nu) \pmod{n}\,. \end{aligned} \tag{1.9}$$

Suppose now that d divides n, and write $d' = n/d$. The simple current $J^{d'}$ that will play an important role in the following, has then order d. We call $\varphi \in P_+^k$ a *fixed point of order* m (with respect to $J^{d'}$) whenever m divides d, and d/m is the smallest positive integer for which $J^{d'(d/m)}\varphi = J^{n/m}\varphi = \varphi$. Then the $J^{d'}$-orbit $\{J^{jd'}\varphi\}$ has cardinality d/m. Write $o(\varphi)$ for the order m of φ. Note that any solution $\varphi \in P_+^k(\mathrm{su}(n))$ to $J^{n/m}\varphi = \varphi$ looks like $\varphi = (\overline{\varphi}, \dots, \overline{\varphi})$ (m copies of $\overline{\varphi}$), where $\overline{\varphi} = (\varphi_0; \dots, \varphi_{n/m-1}) \in P_+^{k/m}(\mathrm{su}(n/m))$, and so

$$t(\varphi) = \sum_{j=0}^{m-1} \sum_{i=0}^{n/m-1} (\frac{nj}{m} + i)\varphi_i = \frac{n}{m}\frac{k}{m}\frac{m(m-1)}{2} + m\,\bar{t}(\overline{\varphi})\,. \tag{1.10}$$

For a given n, k and d, $J^{d'}$ will have fixed points of order m, when and only when m divides $\gcd(d,k)$.

For any divisor d of n with $d' = n/d$, define the matrix $M[d']$ by

$$M[d']_{\lambda\mu} = \sum_{j=1}^{d} \delta_d\left(t(\lambda) + \frac{d'\,j\,k'}{2}\right)\,\delta^{\mu\, J^{jd'}\lambda}\,, \tag{1.11}$$

where $k' = k+n$ if k and n are odd, and $k' = k$ otherwise. Furthermore, $\delta_y(x) = 0$ if $\frac{x}{y} \in \mathbb{Z}$, and $\delta_y(x) = 1$ otherwise. Then $M[d']$ is a modular invariant if and only if the product $(n-1)\,k\,d'$ is even [16]. For instance, $M[n] = I$ is a modular invariant for any $\mathrm{su}(n)$ level k; for su(2), $M[1]$ is a modular invariant if and only if k is even. The modular invariant $M[d']$ corresponds to the WZW model on the non-simply connected group $\mathrm{SU}(n)/\mathbb{Z}_d$ [11,12]. For example, $M[1]$ for su(2) at even level k corresponds to the WZW model with group SO(3).

1.2. D-branes

Next we need to describe the untwisted D-branes, and in particular, their NIM-reps. In the following we shall consider $\mathrm{SU}(n)/\mathbb{Z}_d$, where d divides n. Suppose the level k is fixed, and write $d' = n/d$ as before. In this subsection we assume for simplicity that $M[d']$ defines a modular invariant, *i.e.* that $(n+1)\,k\,d'$ is even.

We need to know the NIM-rep coefficients $\mathcal{N}_{\lambda a}{}^{b}$ corresponding to (1.11). The easiest way to find these is through the formula

$$\mathcal{N}_{\lambda a}{}^{b} = \sum_{\mu\in\mathcal{E}} \frac{\psi_{a\mu} S_{\lambda\mu} \psi^{*}_{b\mu}}{S_{0\mu}} \,, \tag{1.12}$$

where S is the Kac-Peterson modular S matrix, and $\mu \in \mathcal{E}$ are the *exponents*. The general relation of ψ with the modular S matrix for simple current extensions is discussed in [18]; an expression for that S is given in [19]. The resulting general explicit expression for ψ is too awkward to be directly helpful in (1.12); we will give special instances of it throughout this paper.

The fusion ring for SU(n) at level k is generated by the fundamental weights $\Lambda_1, \ldots, \Lambda_{n-1}$. Since the NIM-rep is a representation of the fusion ring, it suffices to consider in (1.1) λ equal to these fundamental weights Λ_i.

The exponents $\mu \in \mathcal{E}$ are the weights with $M[d']_{\mu\mu} \neq 0$. Write $f = \gcd(d,k)$. Then any order $o(\mu)$ must divide f. For the following it is useful to distinguish two cases:

Case A: Either n or k is odd, or either n/d or k/f is even. Then $\mu \in P^k_+$ is an exponent if and only if d divides $t(\mu)$. Such a μ will have multiplicity $o(\mu)$.

Case B: Both n and k are even, and both n/d and k/f are odd. Then the exponents come in two versions: either $f/o(\mu)$ is even and d divides $t(\mu)$ (in which case the multiplicity of μ is $o(\mu)$); or $f/o(\mu)$ is odd and $d/2$ divides $t(\mu)$ (in which case the multiplicity of μ is $o(\mu)/2$).

We will write 'mult(μ)' for this multiplicity (which will either be $o(\mu)$ or $o(\mu)/2$), and let (μ, i), $1 \leq i \leq \text{mult}(\mu)$, denote these exponents. When $\text{mult}(\mu) = 1$, then we shall usually abbreviate $(\mu, 1)$ with μ. Write $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_{d/2}$ where $\mathcal{E}_i$ consists of all exponents μ with $t(\mu) = i \pmod{d}$. Then $\mathcal{E} = \mathcal{E}_0$ if and only if we are in Case A.

The boundary labels $a \in \mathcal{B}$ here correspond to pairs $([\nu], i)$, where $[\nu] = \{J^{jd'}\nu\}$ is the $J^{d'}$-orbit of any weight $\nu \in P^k_+$, and where $1 \leq i \leq o(\nu)$. To simplify notation, we will write $[\nu, i]$ for $([\nu], i)$, and when ν has order $o(\nu) = 1$, then we shall usually write $[\nu]$ instead of $([\nu], 1) = [\nu, 1]$.

Most entries of ψ are easy to compute. When either μ or ν are not fixed points of $J^{d'}$, we find respectively

$$\psi_{[\nu,i]\,\mu} = \frac{\sqrt{d}}{o(\nu)}\, S_{\nu\,\mu} \,, \tag{1.13}$$

$$\psi_{[\nu]\,(\mu,i)} = \begin{cases} \sqrt{\dfrac{d}{\mathrm{mult}(\mu)}}\, S_{\nu\,\mu} & \text{if } \mu \in \mathcal{E}_0 \\ 0 & \text{if } \mu \in \mathcal{E}_{d/2}\,, \end{cases} \tag{1.14}$$

for all i, where (1.13) applies only if $o(\mu) = 1$ (recall that $\mathrm{mult}(\mu)$ can equal $o(\mu)/2$ in Case B). More generally, we obtain

$$\sum_{i=1}^{o(\nu)} \psi_{[\nu,i]\,(\mu,j)} = \begin{cases} \sqrt{\dfrac{d}{\mathrm{mult}(\mu)}}\, S_{\nu\,\mu} & \text{if } \mu \in \mathcal{E}_0 \\ 0 & \text{if } \mu \in \mathcal{E}_{d/2}\,, \end{cases} \tag{1.15}$$

$$\sum_{j=1}^{\mathrm{mult}(\mu)} \psi_{[\nu,i]\,(\mu,j)} = \begin{cases} \dfrac{\sqrt{\mathrm{mult}(\mu) d}}{o(\nu)}\, S_{\nu\,\mu} & \text{if } \mu \in \mathcal{E}_0 \\ 0 & \text{if } \mu \in \mathcal{E}_{d/2}\,, \end{cases} \tag{1.16}$$

for any boundary label $[\nu, i]$ and exponent (μ, j).

Now suppose ν is not a fixed point. We compute from (1.14) and (1.16) that

$$\mathcal{N}_{\lambda\,[\nu]}{}^{[\nu',i]} = \sum_{\mu\in\mathcal{E}_0} \sqrt{\frac{d}{\mathrm{mult}(\mu)}} S_{\nu\mu} \frac{S_{\lambda\mu}}{S_{0\mu}} \frac{\sqrt{\mathrm{mult}(\mu) d}}{o(\nu')} S^*_{\nu'\mu} = \sum_{j=1}^{d/o(\nu')} N_{\lambda\,\nu}{}^{J^{d' j}\nu'}\,, \tag{1.17}$$

for any weight $\lambda \in P_+^k$ and boundary label $[\nu', i]$. More generally, for any weight $\lambda \in P_+^k$ and boundary labels $[\nu, i], [\nu', i']$, we obtain

$$\sum_{i=1}^{o(\nu)} \mathcal{N}_{\lambda\,[\nu,i]}{}^{[\nu',i']} = \sum_{j=1}^{d/o(\nu')} N_{\lambda\,\nu}{}^{J^{d' j}\nu'}\,. \tag{1.18}$$

This does not give all NIM-rep coefficients, but it provides a very useful constraint that can be exploited in general (see section 3). For more specific examples (that will be discussed in section 4) one can give simple explicit formulae for all NIM-rep coefficients. In these cases one can then give a fairly complete description of the D-brane charges. More generally [20], each NIM-rep coefficient $\mathcal{N}_{\lambda\,[\nu,i]}{}^{[\nu',i']}$ can always be expressed in terms of various fusions for $\mathrm{SU}(n/m)$ level k/m, for some m dividing $f = \gcd(d,k)$.

2. The construction for $\mathrm{SU}(3)/\mathbb{Z}_3$ and $SU(2)/\mathbb{Z}_2$

In order to illustrate our general results, which we shall give in sections 3 and 4, let us first discuss the analysis for the case of $\mathrm{SU}(3)/\mathbb{Z}_3$ and $\mathrm{SU}(2)/\mathbb{Z}_2$ in detail.

2.1. The SU(3) situation without fixed points

For $\mathrm{SU}(3)/\mathbb{Z}_3$, the situation is simplest if the level k is not divisible by 3 since then the orbifold does not have any fixed points. (The situation when 3 divides k will be discussed in the next subsection.) In this case, (1.11) implies that the spectrum of the corresponding WZW model is given by

$$\mathcal{H} = \bigoplus_{t(\lambda)=0 \,(\mathrm{mod}\ 3)} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}^*_\lambda \oplus \bigoplus_{t(\lambda)=1 \,(\mathrm{mod}\ 3)} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}^*_{J\lambda} \oplus \bigoplus_{t(\lambda)=2 \,(\mathrm{mod}\ 3)} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}^*_{J^2\lambda}\,, \quad (2.1)$$

where $\lambda \in P^k_+(\mathrm{su}(3))$ is an allowed highest weight for $\widehat{\mathrm{su}}(3)$ at level k, *i.e.* $\lambda = (\lambda_1, \lambda_2)$ satisfies $\lambda_1 + \lambda_2 \le k$. The quantity $t(\lambda)$ is defined by

$$t(\lambda) = \lambda_1 + 2\lambda_2\,, \quad (2.2)$$

and J is the simple current that acts on allowed highest weights as

$$J(\lambda_1, \lambda_2) = (k - \lambda_1 - \lambda_2, \lambda_1)\,. \quad (2.3)$$

Since k here is not divisible by 3, none of the weights $\lambda \in P^k_+(\mathrm{su}(3))$ are invariant under J.

The boundary states of this theory are labelled by the J-orbits in $P^k_+(\mathrm{su}(3))$, and are denoted by $[a]$, where $a \in P^k_+(\mathrm{su}(3))$. More explicitly, $[a] = [b]$ if and only if $a = J^l b$, for some l. The corresponding NIM-rep is simply given by

$$\mathcal{N}_{\lambda [a]}{}^{[b]} = \sum_{i=0}^{2} N_{\lambda a}{}^{J^i b}\,, \quad (2.4)$$

where N denotes the fusion rules of $\widehat{\mathrm{su}}(3)$ at level k. We now make the ansatz that the charge of the D-brane corresponding to $[a]$ is given by

$$q_{[a]} = \dim_{\mathrm{su}(3)}(a) = \frac{(a_1+1)(a_2+1)(a_1+a_2+2)}{2}\,. \quad (2.5)$$

This definition is well-defined modulo $M_{\mathrm{SU}(3)}$,

$$M_{\mathrm{SU}(3)} = \frac{k+3}{\gcd(k+3, 2)}\,, \quad (2.6)$$

since

$$\dim(a) = \dim(Ja) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)})\,, \quad (2.7)$$

as was already observed in [5]. By construction of $M_{\mathrm{SU}(3)}$ (see (1.2)), we have

$$\dim(\lambda)\,\dim(a) = \sum_{b\in P_+^k(\mathrm{su}(3))} N_{\lambda a}{}^{b}\,\dim(b) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)})\,. \tag{2.8}$$

It then follows that the right hand side of (1.1) equals

$$\begin{aligned}\sum_{[b]} \mathcal{N}_{\lambda[a]}{}^{[b]}\, q_{[b]} &= \sum_{[b]}\sum_{i=0}^{2} N_{\lambda a}{}^{J^i b}\,\dim(b) \\ &= \sum_{[b]}\sum_{i=0}^{2} N_{\lambda a}{}^{J^i b}\,\dim(J^i b) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)}) \\ &= \sum_{b\in P_+^k(\mathrm{su}(3))} N_{\lambda a}{}^{b}\,\dim(b) \\ &= \dim(\lambda)\, q_{[a]} \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)})\,.\end{aligned} \tag{2.9}$$

Thus the ansatz (2.5) solves (1.1) with $M = M_{\mathrm{SU}(3)}$. It is unique (as we explain in the next section), so the twisted charges $q^t_{[a]}$ are all 0, with $K = \mathbb{Z}_M$.

2.2. The SU(3) situation with fixed points

The analysis is more complicated when k is divisible by 3, since $P_+^k(\mathrm{su}(3))$ contains then a highest weight $\phi = (k/3, k/3)$ for which $J\phi = \phi$. In this case the spectrum of the theory is given by

$$\mathcal{H} = \bigoplus_{[\lambda],t(\lambda)=0\,(\mathrm{mod}\ 3)} \Big(\mathcal{H}_\lambda \oplus \mathcal{H}_{J\lambda} \oplus \mathcal{H}_{J^2\lambda}\Big) \otimes \Big(\bar{\mathcal{H}}^*_\lambda \oplus \bar{\mathcal{H}}^*_{J\lambda} \oplus \bar{\mathcal{H}}^*_{J^2\lambda}\Big) \oplus 3\,\mathcal{H}_\phi \otimes \bar{\mathcal{H}}_\phi\,, \tag{2.10}$$

where the first sum is over all J-orbits $[\lambda]$ with $\lambda \neq \phi$ for which $t(\lambda) = 0$ (mod 3). As before, some boundary states are labelled by the J-orbits $[a]$ with $a \neq \phi$, but now there are in addition 3 boundary states $[\phi, i]$ with $i = 1, 2, 3$ associated to the fixed point ϕ. In addition to the NIM-rep coefficients (2.4) we now have [21]

$$\mathcal{N}_{\lambda[a]}{}^{[\phi,i]} = N_{\lambda a}{}^{\phi} \tag{2.11}$$

as well as

$$\mathcal{N}_{\lambda[\phi,i]}{}^{[\phi,j]} = \begin{cases} \frac{1}{3} N_{\lambda\phi}{}^{\phi} & \text{if } N_{\lambda\phi}{}^{\phi} = 0\ (\mathrm{mod}\ 3) \\ \frac{1}{3}(N_{\lambda\phi}{}^{\phi} - 1) + \delta^{ij} & \text{if } N_{\lambda\phi}{}^{\phi} = 1\ (\mathrm{mod}\ 3) \\ \frac{1}{3}(N_{\lambda\phi}{}^{\phi} + 1) - \delta^{ij} & \text{if } N_{\lambda\phi}{}^{\phi} = 2\ (\mathrm{mod}\ 3). \end{cases} \tag{2.12}$$

For the following it is useful to note that $N_{\lambda\phi}{}^{\phi} = 0$ if either $\lambda_1 = 2 \pmod 3$ or $\lambda_1 \neq \lambda_2 \pmod 3$.

We now want to construct three independent solutions to (1.1). The first solution is associated to the 'untwisted' D0-brane charge. To this end we define as before

$$q_{[a]} = \dim(a)\,. \tag{2.13}$$

In order to analyse whether this ansatz (together with a suitable formula for $q_{[\phi,i]}$ that we shall deduce in the following) solves (1.1), we observe that it is sufficient to check (1.1) for λ equalling one of the two fundamental weights $(1,0),(0,1)$ of su(3). Let us first consider (1.1) for the situation where a denotes a boundary $[a]$ with $a \neq \phi$. Then the analysis of the previous subsection shows that the above charges solve (1.1) modulo $M_{\mathrm{SU}(3)}$ provided that

$$\sum_{i=1}^{3} q_{[\phi,i]} = \dim(\phi) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)})\,. \tag{2.14}$$

On the other hand if $a = [\phi, i]$, (1.1) is solved for $\lambda = (1,0)$ modulo $M_{\mathrm{SU}(3)}$ provided that

$$3\, q_{[\phi,i]} = \dim((k/3+1, k/3)) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(3)})\,, \tag{2.15}$$

with an equivalent relation for $\lambda = (0,1)$. We can solve (2.15) by defining

$$q_{[\phi,i]} = \frac{1}{3}\dim((k/3+1,k/3)) + l_i \frac{M_{\mathrm{SU}(3)}}{3}\,, \tag{2.16}$$

where $l_i \in \mathbb{Z}$ is defined modulo 3 (given that $q_{[\phi,i]}$ is only defined modulo $M_{\mathrm{SU}(3)}$). Since $\dim((a,b)) = (a+1)(b+1)(a+b+2)/2$, and 3 divides k, the right hand side of (2.16) is indeed an integer. It therefore only remains to check whether this ansatz solves also (2.14). Using the above dimension formula one finds that this is the case provided that

$$\sum_{i=1}^{3} l_i = (-1)^k \qquad (\mathrm{mod}\ 3)\,. \tag{2.17}$$

It is always possible to choose the l_i in this manner, and thus we have shown that (2.13) and (2.16) define a solution to (1.1) with $M = M_{\mathrm{SU}(3)}$.

It is clear from the above discussion that this solution is not unique (since only the sum of the l_i is determined). If we subtract two consistent solutions from one another, we obviously obtain another consistent solution. The resulting solution can be interpreted as

measuring the 'twisted' charges of the orbifold since $q^t_{[a]} = 0$ for all non-fixed points a. If we make this ansatz, the analogue of (2.15) now implies that

$$3\, q^t_{[\phi,i]} = 0 \qquad (\mathrm{mod}\ M^t)\,, \tag{2.18}$$

and thus that $M^t = 3$. Furthermore, the analogue of (2.14) gives that

$$\sum_{i=1}^{3} q^t_{[\phi,i]} = 0 \qquad (\mathrm{mod}\ 3)\,. \tag{2.19}$$

It is clear that there are two independent solutions to these equations: we can choose $q^t_{[\phi,1]}$ and $q^t_{[\phi,2]}$ independently, and $q^t_{[\phi,3]}$ is then determined by (2.19). Thus the charge vectors are 3-dimensional in this case, and the total charge group is

$$K = \mathbb{Z}_{M_{\mathrm{SU}(3)}} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3\,. \tag{2.20}$$

2.3. The example of SO(3)

The final phenomena which can occur is well-illustrated by SO(3), the simplest example of a non-simply connected Lie group. Let us recall from the work of [22,11,12] that the SO(3) WZW model only exists for even k, and that its spectrum is given by

$$\mathcal{H} = \bigoplus_{j=0}^{k/2} \mathcal{H}_{2j} \otimes \bar{\mathcal{H}}_{2j} \oplus \bigoplus_{j=1}^{k/2} \mathcal{H}_{2j-1} \otimes \bar{\mathcal{H}}_{k+1-2j}\,, \tag{2.21}$$

provided that 4 does not divide k (what we call 'Case B' in section 1.2), and by

$$\mathcal{H} = \bigoplus_{j=0}^{k/4-1} \Big(\mathcal{H}_{2j} \oplus \mathcal{H}_{k-2j}\Big) \otimes \Big(\bar{\mathcal{H}}_{2j} \oplus \bar{\mathcal{H}}_{k-2j}\Big) \oplus 2\,\mathcal{H}_{k/2} \otimes \bar{\mathcal{H}}_{k/2}\,, \tag{2.22}$$

provided that 4 does divide k ('Case A'). Here $\mathcal{H}_j$ is the highest weight representation of the affine $\widehat{\mathrm{su}}(2)$ algebra at level k with highest weight j; we choose the convention that j is integral so that the vector representations of su(2) are described by even j, while the spinor representations correspond to odd j.

It is easy to read off from (2.21) and (2.22) that the theory has $n = k/2+2$ exponents, *i.e.* that there are n different (untwisted) Ishibashi states, and hence also n different (untwisted) boundary states. For the usual reason, it is enough to consider the 2-dimensional fundamental representation of su(2). Its NIM-rep matrix $\mathcal{N}_1$ is given by the D_n Dynkin diagram [23] (the D-branes of this theory were first constructed in a series of papers in [24,25,26]):

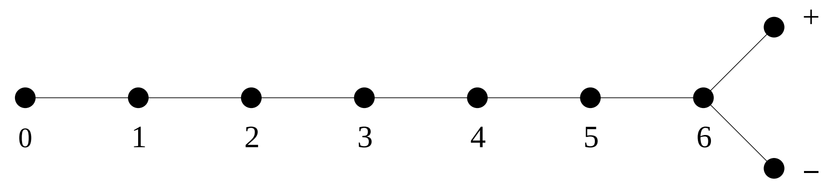

Fig. 1: The NIM-rep graph for su(2) corresponding to the SO(3) modular invariant D_n for $n = k/2 + 2 = 9$.

Let us label the boundary states starting from the left by $a = 0, \ldots, n-3$, with the two boundary states at the tail being denoted by $a = \pm$. Then the charges, which we shall denote as q_l, $l = 0, \ldots, n-3$ as well as $q_\pm$, have to satisfy the relations

$$\begin{aligned} 2\,q_0 &= q_1 \\ 2\,q_l &= q_{l-1} + q_{l+1} \qquad\qquad l = 1, \ldots, n-4 \\ 2\,q_{n-3} &= q_{n-4} + q_+ + q_- \\ 2\,q_+ &= q_{n-3} \\ 2\,q_- &= q_{n-3}\,. \end{aligned} \tag{2.23}$$

The first two equations imply that

$$q_l = (l+1)\,q_0\,, \qquad l = 0, \ldots, n-3\,. \tag{2.24}$$

The third last equation then gives that

$$q_+ + q_- = (n-1)\,q_0\,, \tag{2.25}$$

while the sum of the last two equations is $2(q_+ + q_-) = 2(n-2)q_0$. By subtracting this equation from twice (2.25), it follows that

$$2\,q_0 = 0 \qquad (\mathrm{mod}\ M)\,. \tag{2.26}$$

For the untwisted charge component, $q_0 = 1$, and $M = 2$. If k is divisible by 4, n is even, and $q_+ + q_- = 1$, as well as $2q_+ = 2q_- = 0$ mod 2. Thus the untwisted charge solution is

$$k \text{ divisible by 4:} \qquad q_l = \begin{cases} 1 & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \qquad q_+ = 1\,, \qquad q_- = 0\,. \tag{2.27}$$

This charge assignment is not unique since we could have also chosen $q_+ = 0$ and $q_- = 1$. Thus we have a second (twisted) charge solution with $M^t = 2$ and

$$k \text{ divisible by 4:} \qquad q^t_l = 0\,, \qquad q^t_+ = 1\,, \qquad q^t_- = 1\,. \tag{2.28}$$

The corresponding charge group is therefore

$$k \text{ divisible by 4:} \qquad K = \mathbb{Z}_2 \oplus \mathbb{Z}_2\,. \tag{2.29}$$

On the other hand, if k is not divisible by 4, n is odd, and $2q_+ = 2q_- = q_0$ mod 2 with $q_+ + q_- = 0$ mod 2. In this case the untwisted solution is (with $M = 2$)

$$k \text{ not divisible by 4:} \qquad q_l = \begin{cases} 1 & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \qquad q_+ = \frac{1}{2} \qquad q_- = -\frac{1}{2}\,. \tag{2.30}$$

Note that $3q_l = q_l$, and that $3q_+ = -\frac{1}{2}$ as well as $3q_- = \frac{1}{2}$. Thus this solution encompasses both solutions to $q_+ + q_- = 0$, and there is therefore no separate twisted solution — *i.e.* the twisted solution (also given by (2.28) and corresponding to $M^t = 2$) is redundant. Since we now have half-integer charges in (2.30), the resulting charge group is in this case

$$k \text{ not divisible by 4:} \qquad K = \mathbb{Z}_4\,. \tag{2.31}$$

The fractional charges (2.30) require that the charge group K in (2.31) be formed from the group extension of $\mathbb{Z}_M = \mathbb{Z}_2$ by $\mathbb{Z}_{M^t} = \mathbb{Z}_2$, rather than their direct sum as in (2.29). Note that if k is divisible by 4, the modular invariant is of Case A, while if k is not divisible by 4, we are in Case B.

This is different to what one may have expected based on the analysis of [27], or what seems to be suggested in [28]. In particular, the above charge groups do not grow with k. This result has also a simple quasi-geometric interpretation. As is explained in [29,28,30] (see also [31]), the brane corresponding to spin l for SO(3) is the $\mathbb{Z}_2$ invariant superposition of the SU(2) brane corresponding to l, and the SU(2) brane corresponding to $k - l$. The D0-brane charge of these two branes are $q_l = l + 1$ and $q_{k-l} = k - l + 1$, respectively. The configuration that appears in SO(3) therefore has total D0-brane charge $q = l+1+k-l+1 = k+2$, and this vanishes since the charge group of SU(2) is $\mathbb{Z}_{k+2}$. The only D-branes that carry untwisted D0-brane charge are therefore the branes labelled by $a = \pm$, that correspond to the $\mathbb{Z}_2$-invariant 2-cycle. These branes carry the same untwisted D-brane charge (namely $(k + 2)/2$). Thus the D-branes of SO(3) only see a $\mathbb{Z}_2$ subgroup

of the untwisted D-brane charge group. (Confusingly, this is what is measured by the 'twisted' charge solutions above!)

More formally, the superpositions of boundary states that are relevant for SO(3) are related by the action of the simple current J. In SU(2), the charges of the brane associated to a representation j is simply $\dim(j) = j+1$, and the SO(3) invariant configurations do not carry any charge since

$$\dim(Jj) = -\dim(j) \qquad (\mathrm{mod}\ M_{\mathrm{SU}(2)})\,. \tag{2.32}$$

In fact, since J is a simple current, this property already follows from the fact that $\dim(J0) = -1$ (mod $M_{\mathrm{SU}(2)}$).

Incidentally, it is intriguing that the charge groups for SO(3) level k match exactly the centre of the associated D_n diagram (here $n = k/2+2$).* Likewise, the diagonal SU(2) level k theory has charge group $\mathbb{Z}_{k+2}$, which matches the centre of the associated A_n diagram (here $n = k+1$). Easy calculations confirm that the SU(2) exceptional models associated to E_6, E_7 and E_8 have charge groups $\mathbb{Z}_3$, $\mathbb{Z}_2$ and $\mathbb{Z}_1$, respectively, which again equal their centres. We do not have an explanation for this exact matching between charge groups and centres for the SU(2) theories, which seems to deepen the mysterious A-D-E correspondence here.

3. General results

Let us now study the D-brane charges and the charge groups in the general case whose NIM-rep coefficients were at least partially described in section 1.2.

Recall from the introduction that any solution to (1.1) must have order dividing $M_{\mathrm{SU}(n)}$, so in particular both the order of the untwisted and twisted solution, M^u and M^t, must divide $M_{\mathrm{SU}(n)}$. This follows by the same argument given in [8], since the NIM-rep here is again a representation of the fusion ring.

First we want to analyse in which situations the untwisted charge group is pathologically small, as was the case for SO(3) above.

* We thank Mark Walton for pointing this out to us.

3.1. Characterising the pathological cases

As for any simple current, $\mathcal{N}_{J^{d'}0}$ will be a permutation matrix. From (1.18) with $\lambda = J^{d'}0$, we see that $\mathcal{N}_{J^{d'}0,[\nu,i]}{}^{[\nu',i']} = 0$ unless the $J^{d'}$-orbits $\langle J^{d'}\rangle\nu$ and $\langle J^{d'}\rangle\nu'$ are equal. Thus

$$\mathcal{N}_{J^{d'}0,[\nu,i]}{}^{[\nu',i']} = \delta_{\nu,\nu'}\delta_{i',\pi_\nu i}\,, \tag{3.1}$$

where π_ν is a permutation of the indices $\{1,\ldots,o(\nu)\}$. Putting $\lambda = J^{d'}0$ and $a = [0]$ into (1.1), we obtain the important condition that for the untwisted solution with $q_{[0]} = 1$

$$\dim(J^{d'}0) = 1 \qquad (\mathrm{mod}\ M)\,. \tag{3.2}$$

[Note that the integer M that appears in (3.2) need not be the order M^u of the corresponding solution: we have normalised the untwisted solution so that $q_{[0]} = 1$, but fractional charges are possible for fixed points, as was the case for the example of SO(3) that was considered in section 2.3.]

Now, the simple current J of su(n) level k obeys

$$\dim(J0) = \prod_{j=1}^{n-1} \frac{k+n-j}{j} = (-1)^{n+1} \qquad (\mathrm{mod}\ M_{\mathrm{SU}(n)}) \tag{3.3}$$

and thus

$$\dim(J^{d'}0) = (-1)^{(n+1)d'} \qquad (\mathrm{mod}\ M_{\mathrm{SU}(n)})\,. \tag{3.4}$$

It therefore follows that M can be at most 2 if $(n+1)d'$ is odd. The charge group for these theories will therefore be pathologically small (just as for SO(3)), and we shall sometimes refer to them as being *pathological*. Incidentally, the theories for which $(n+1)d'$ is odd are precisely those for which the partition function $M[d']$ is only modular invariant for k even. From a geometrical point of view this constraint on k reflects some obstruction in the definition of the Wess-Zumino term [12].

We observe that Case B always corresponds to $(n+1)d'$ odd, and will therefore always exhibit the above pathological behaviour. On the other hand, Case A may or may not be pathological, and it can only be pathological if d is even.

We can say more, by studying $\mathcal{N}_{J^{d'}0}$ further. Consider first Case A in section 1.2, *i.e.* either n or k is odd, or either n/d or k/f is even. Then one can show (by considering their eigenvalues $S_{J^{d'}0\,\mu}/S_{0\,\mu}$) that in fact the matrix $\mathcal{N}_{J^{d'}0}$ is the identity matrix. Thus in Case A, any charge solution (untwisted or twisted) can only be satisfied mod M, where

$$\dim(J^{d'}0) = 1 \qquad (\mathrm{mod}\ M)\,, \tag{3.5}$$

and $M = M^u$ or $M = M^t$. So in particular, if Case A is pathological, any solution to (1.1) has order at most 2, and thus the total charge group is a direct sum of $\mathbb{Z}_2$'s. (An example of this situation is SO(3) for k divisible by 4.)

In Case B, the eigenvalues $S_{J^{d'}0\,\mu}/S_{0\,\mu}$ are all ± 1, and therefore $\mathcal{N}_{J^{d'}0}$ necessarily has order 2. By counting -1's, we find that in (3.1) $\pi_\nu(i) = i$ for all boundary labels $[\nu, i]$, except for a number exactly equal to the number of boundary labels $[\nu, i]$ for which $f/o(\nu)$ is odd and divides $\bar{t}(\overline{\nu})$. Thus it can be expected (but we have not proven this) that the permutation π_ν is nontrivial for precisely those ν and no others, in which case π_ν is order 2 without fixed points. In any case, (3.5) will not hold in general in Case B. (For example, it does not hold for the case of SO(3) when 4 does not divide k.)

For completeness we also give the corresponding simple current dimensions for all other groups:

For B_n: $\dim(J0) = -1 \bmod M_B$ [8];

For C_n: $\dim(J0) = (-1)^{n(n+1)/2} \bmod M_C$;

For D_n: $\dim(J_v 0) = 1$, and $\dim(J_s 0) = \dim(J_c 0) = (-1)^{n(n-1)/2}$ all taken mod M_D;

For E_6: $\dim(J0) = 1 \bmod M_{E_6}$; and finally

For E_7: $\dim(J0) = -1 \bmod M_{E_7}$.

It follows from these results, that for example the D-brane charges of $\mathrm{B}_n/\mathbb{Z}_2$ and $\mathrm{E}_7/\mathbb{Z}_2$ will be pathologically small, and similarly for C_n (provided that $n(n+1)/2$ is odd) and D_n (provided that $n(n-1)/2$ is odd and the quotient group involves J_c or J_s).

3.2. The complete solution for k coprime to d

The untwisted solution is characterised by setting $q_{[0]} = 1$. Then (1.17) and (1.1) with $a = [0]$ require

$$q_{[\lambda]} = \dim(\lambda) \tag{3.6}$$

for any non-fixed point $\lambda \in P_+^k$ of $J^{d'}$ (*i.e.* $o(\lambda) = 1$). If there is more than one solution for which (3.6) is satisfied, we can consider their difference, which defines then a 'twisted' solution q_a^t with

$$q_{[\lambda]}^t = 0\,, \tag{3.7}$$

for any λ that is not a fixed point of $J^{d'}$. Note that these results apply regardless of the value of M or M^t. More generally, we find from (1.18) the conditions

$$\sum_{i=1}^{o(\lambda)} q_{[\lambda,i]} = \dim(\lambda) \tag{3.8}$$

and

$$\sum_{i=1}^{o(\lambda)} q^t_{[\lambda,i]} = 0\,, \tag{3.9}$$

where $\lambda \in P^k_+$ is any weight.

We will use these equations in the following sections. For now we observe that *they provide an immediate and complete solution to the situation where there are no fixed points.* This occurs precisely when $\gcd(k,d)=1$. This solution is given by (3.6) with $M = M_{\mathrm{SU}(n)}$ (if $(n+1)d'$ is even) or $M = \gcd(2, M_{\mathrm{SU}(n)})$ (if $(n+1)d'$ is odd). It is the unique solution, in the sense that any other solution is a multiple of this one. For this M, (3.6) is well-defined, and (1.1) follows from (1.2). On the other hand, the twisted charges must all vanish. Thus the charge group here is $K = \mathbb{Z}_M$. This analysis also applies directly to the other affine algebras.

The condition $\gcd(k,d)=1$ is satisfied for example for SU(2) when the level k is odd. The NIM-rep is then given by the 'tadpole graph' of [23]. As is well known, this does not define a modular invariant (since k has to be even for the SO(3) theory) and $M=1$ here (*i.e.* all charges, twisted and untwisted, are trivial). The same will happen whenever $M[d']$ is not a modular invariant. In a sense this is the reason why the D-brane charges for SO(3) behave rather atypically: the simple general solution never arises for the SO(3) theory. (The actual analysis for SO(3) was described in detail in the last section.) On the other hand, there are obviously many modular invariants for which the condition $\gcd(k,d)=1$ is satisfied.

3.3. Some generalisations

We can generalise this construction as follows. For simplicity assume that $(n+1)d'$ is even so that $\dim(J^{d'}0) = 1 \bmod M_{\mathrm{SU}(n)}$. (Otherwise, as is explained in section 3.1, the charge group is anyway pathologically small.) We claim that a (in general fractional) solution to (1.1), taken modulo $\widehat{M} = M/\gcd(d,M)$, where $M = M_{\mathrm{SU}(n)}$, is given by

$$q_{[\mu,i]} = \frac{\dim(\mu)}{o(\mu)}\,, \tag{3.10}$$

for all $1 \le i \le o(\mu)$ and all weights $\mu \in P^k_+$. (Note that this solution reduces to (3.6) when μ is not a fixed point.) Verifying that (3.10) does indeed satisfy (1.1) amounts to showing that

$$\dim(\lambda)\dim(\mu) = o(\mu)\sum_{[\nu]}\sum_{i=1}^{o(\nu)} \mathcal{N}_{\lambda\,[\mu,j]}{}^{[\nu,i]}\,\frac{\dim(\nu)}{o(\nu)} \qquad \Big(\mathrm{mod}\ \gcd(M, o(\mu)\widehat{M})\Big)\,. \tag{3.11}$$

In fact, we can show that this congruence holds mod M: by (1.18) (or rather the transpose of this equation), the right-hand side becomes

$$o(\mu)\sum_{[\nu]}\sum_{j=1}^{d/o(\mu)} N_{\lambda\, J^{d'j}\mu}{}^{\nu}\,\frac{\dim(\nu)}{o(\nu)} = \sum_{[\nu]}\sum_{j=1}^{d/o(\nu)} N_{\lambda\,\mu}{}^{J^{d'j}\nu}\dim(\nu) = \sum_{\nu} N_{\lambda\,\mu}{}^{\nu}\dim(\nu) \pmod{M}\,, \tag{3.12}$$

using (1.9) twice, as well as the fact that $\dim(J^{d'j}\nu) = \dim(\nu)$ mod M. Of course, (1.2) then implies that the right-side of (3.12) equals the left-side of (3.11) mod M.

Now, a very common situation is when d is coprime to $M_{\mathrm{SU}(n)}$. This happens precisely when, for each prime p dividing d, the largest power p^β dividing $k+n$, obeys $p^\beta < n$. In this case, d is invertible mod $M_{\mathrm{SU}(n)}$ and since $o(\mu)$ divides d, the charges (3.10) are manifestly integers. The resulting charge for $[0]$ equals then $q_{[0]} = 1$, and (3.10) defines an untwisted solution mod $M_{\mathrm{SU}(n)}$, the maximal possible value.

However, when d and $M_{\mathrm{SU}(n)}$ are not coprime, this solution will not describe the most general untwisted solution (up to the ambiguity described by a twisted solution), because rescaling it by d leads to $q_{[0]} = \gcd(d, M_{\mathrm{SU}(n)})$, not $q_{[0]} = 1$. For example, for the case of SU(3), multiplying (2.13) and (2.16) by 3 and subtracting it from this rescaled solution yields one of our twisted solutions. Thus (for $\mathrm{SU}(3)/\mathbb{Z}_3$ when 3 divides k) the solution (3.10) is contained in (2.13) and (2.16) (together with the twisted solutions) but not vice versa. More generally, (3.10) together with all of the twisted solutions, will probably only generate a space of solutions to (1.1) which is of index $\gcd(d, M_{\mathrm{SU}(n)})$ in the complete space of solutions. It seems reasonably straightforward, given a specific group $\mathrm{SU}(n)/\mathbb{Z}_d$, to find the missing solutions (we shall find these for other general classes in the next section), but we do not know a general expression for them. For certain classes of theories we can however be more specific. This will be described in the following section.

4. The analysis for $\mathrm{SU}(n)/\mathbb{Z}_d$, d prime

Until section 4.4, we shall assume that d is any prime dividing n. If $d > 2$ then the theory is always not pathological (*i.e.* $(n+1)d'$ is even); the analysis of the previous section then shows that the untwisted solution satisfies (1.1) modulo M, where M is at least $M = M_{\mathrm{SU}(n)}/\gcd(d, M_{\mathrm{SU}(n)})$ and probably equal to $M = M_{\mathrm{SU}(n)}$. In this section we shall give further evidence that the untwisted solution actually solves (1.1) modulo $M = M_{\mathrm{SU}(n)}$. We shall also give a complete description of the twisted charges (for all values of k) in this case; this is described in section 4.2.

The situation is different if $d=2$, since the theory may then be pathological (this happens precisely if $n/2$ is odd). If the theory is not pathological the same arguments as for $d>2$ apply, and we can give a complete description of the twisted charges and provide good evidence that the untwisted solution satisfies (1.1) modulo $M_{\mathrm{SU}(n)}$. In the pathological case, the situation depends on whether we are in Case A or Case B. In the former situation, the charge group is either trivial or $K=\mathbb{Z}_2\oplus\mathbb{Z}_2$, while in Case B it is either $\mathbb{Z}_4$, $\mathbb{Z}_2\oplus\mathbb{Z}_2$, $\mathbb{Z}_2$, or trivial. This is explained in section 4.3

In section 4.4 we briefly discuss the similarities and differences which occur when d is allowed to be composite. This helps to motivate the conjectures we give in the concluding section.

To begin let us collect some useful facts that will be needed throughout this section.

4.1. Some useful facts

Let α be the highest power of d dividing n. In the following Claim, d is assumed to be any prime.

Claim. (a) The greatest common divisor gcd(dim(λ)), as λ runs over all highest weights in P_+^k with d coprime to $t(\lambda)$, equals d^α.
(b) For any fundamental weight Λ_ℓ of su(n),

$$\dim(\Lambda_\ell)=\begin{cases}\dim(\overline{\Lambda}_{\ell/d}) & \text{if } d \text{ divides } \ell\\ 0 & \text{otherwise}\end{cases}\pmod{d^\alpha}\ ,\tag{4.1}$$

where $\dim(\overline{\Lambda}_{\ell/d})$ is the dimension of the su(d') fundamental weight $\overline{\Lambda}_{\ell/d}$.
(c) Assume d divides k. Then for any fundamental weight Λ_ℓ with d dividing ℓ, and any $J^{d'}$-fixed points $\phi,\varphi\in P_+^k(\mathrm{su}(n))$,

$$N_{\Lambda_\ell\,\phi}{}^{\varphi}=\overline{N}_{\overline{\Lambda}_{\ell/d}\,\overline{\phi}}{}^{\overline{\varphi}}\ ,\tag{4.2}$$

where $\overline{\Lambda}_{\ell/d},\overline{\phi},\overline{\varphi}$ are the obvious weights in $P_+^{k/d}(\mathrm{su}(d'))$.

The equality (4.2) of fusion coefficients can be seen directly from the Pieri rule. The bottom part of (4.1) follows from (a). For the top part of (4.1), write

$$\dim(\Lambda_\ell)=\binom{n}{\ell}=\Big(\prod_{i=0}^{\ell/d-1}\frac{n-di}{\ell-di}\Big)\Big(\prod_{\substack{i=1\\ \gcd(d,i)=1}}^{\ell-1}\frac{n-i}{\ell-i}\Big)\,,\tag{4.3}$$

where we have grouped together all terms with denominator and numerator divisible by d (first factor), and coprime to d (second factor). The first factor manifestly equals $\binom{n/d}{\ell/d}$. It is divisible by $d^{\alpha-\delta}$, where d^δ is the largest power of d dividing ℓ. The numerator and denominator of each fraction $\frac{n-i}{\ell-i}$ in the second factor are manifestly congruent modulo $d^{\min(\alpha,\delta)}$. Thus the second factor is congruent to 1 modulo $d^{\min(\alpha,\delta)}$ (since the denominators are coprime to d, they can be inverted in the ring $\mathbb{Z}_{d^{\min(\alpha,\delta)}}$). Putting this together, we get (b).

Finally, we turn to the proof of (a). We know that the character ch_λ (hence dimension $\dim(\lambda)$) of any weight with $t(\lambda)$ coprime to d, is a polynomial over $\mathbb{Z}$ in the characters $\mathrm{ch}_{\Lambda_\ell}$ (hence dimensions $\dim(\Lambda_\ell)$) of the fundamentals, where each term in that polynomial involves at least one Λ_ℓ with d coprime to ℓ. So it suffices to compute $\gcd(\dim(\Lambda_\ell))$ for ℓ coprime to d. First, this gcd must divide $\dim(\Lambda_1) = n$. If p^m is the maximal power of a prime $p \neq d$ dividing n, then $\dim(\Lambda_{p^m}) = \binom{n}{p^m}$ will be coprime to p. On the other hand, (4.3) shows d^α will divide any $\binom{n}{\ell}$ for ℓ coprime to d. This gives us part (a) of the claim.

4.2. The case when d is an odd prime

Now suppose that d is an odd prime dividing n, and denote as before $d' = n/d$. In the previous section we found the general solution for the case when d and k are coprime, since there are then no fixed points. We now want to address the situation where d and k contain a common factor. Since d is prime, this can only be the case if d divides k. In this case, a fixed point necessarily has order d with respect to $J^{d'}$. Note that the matrix $M[d']$ of (1.11) will then be a simple current modular invariant (*i.e.* $(n+1)kd'$ is even), and (3.2) will be satisfied for $M = M_{\mathrm{SU}(n)}$.

The boundary states are parametrised by all $J^{d'}$-orbits $[\nu]$ in $P_+^k(\mathrm{su}(n))$ where ν is not a fixed point of $J^{d'}$, together with exactly d copies $[\varphi, i]_{1\leq i\leq d}$ for each fixed point φ. These fixed points φ are in natural bijection with the level k/d weights $\overline{\varphi}$ of $\mathrm{su}(d')$, as explained in section 1.1. The exponents consist of all weights μ with d dividing $t(\mu)$ (by (1.10) this includes all $J^{d'}$-fixed points), where non-fixed points μ are counted exactly once and the fixed points again come with multiplicity d: $(\phi, i)_{1\leq i\leq d}$. Although it may not be completely obvious, these two sets (namely boundary labels and exponents) always have the same cardinality.

Most entries of the ψ matrix in (1.12) are given by (1.13) and (1.14). When both $\mu = \phi$ and $\nu = \varphi$ are fixed points, we get

$$\psi_{[\varphi,h]\,(\phi,j)} = \frac{1}{d}\,\left(S_{\varphi\phi} + \overline{S}_{\bar{\varphi}\bar{\phi}}(d\,\delta^{h\,j} - 1)\right)\,, \tag{4.4}$$

where $\overline{S}_{\bar{\varphi}\bar{\phi}}$ is the S-matrix for su(d') at level k/d.

A technical observation makes it possible to proceed with an explicit expression for the associated NIM-rep. *Fixed point factorisation* [32] explains how to express the su(n) level k S-matrix entries $S_{\lambda\phi}$ involving $J^{d'}$-fixed points ϕ (λ can be arbitrary) in terms of the su(d') level k/d S-matrix entries involving $\overline{\phi}$. With this observation, it is then possible to calculate the remaining NIM-rep coefficients $\mathcal{N}_{\lambda\, a}{}^{b}$ in terms of WZW fusion rules. Together with the ones that were already given in (1.17), we can now calculate

$$\mathcal{N}_{\Lambda_\ell\,[\phi,i]}{}^{[\varphi,j]} = \begin{cases} \delta^{ij}\,\overline{N}_{\bar{\Lambda}_{\ell/d}\bar{\phi}}{}^{\bar{\varphi}} & \text{if } d \text{ divides } \ell \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

where, as usual, the bar denotes su(d') quantities at level k/d. Analogous expressions can be found for other λ, but they are more complicated.

As mentioned earlier, the NIM-rep is uniquely determined by its values for the fundamental weights $\lambda = \Lambda_\ell$, $1 \leq \ell \leq n/2$, and these (together with the simple currents) are also the most useful. For convenience we also include the simple currents

$$\mathcal{N}_{J^\ell 0\,[\phi,i]}{}^{[\varphi,j]} = \delta_{J^\ell\phi\,\varphi}\,\delta^{ij}\,. \tag{4.6}$$

With these preparations we can now determine the D-brane charges. Recall from the beginning of section 3 that both M^u and M^t divide $M_{\mathrm{SU}(n)}$. Furthermore, as explained above, (3.2) is automatically satisfied.

First let us determine the twisted charges, *i.e.* the solutions $q^t_{[\nu,i]}$ to (1.1) with $q^t_{[0]} = 0$. Then, because of (3.7), $q^t_{[\nu]} = 0$ for any ν that is not a fixed point. Furthermore, we have (3.9). By (1.17), we find that M^t must divide $\dim(\lambda)\, q^t_{[\phi,i]}$, whenever the fusion product of λ and ϕ avoids all fixed points. By (1.9) and (1.10), this will happen for example whenever d does not divide $t(\lambda)$. Because of claim (a) the gcd of the dimensions of these weights is d^α. So M^t must divide $d^\gamma = \gcd(d^\infty, n, M_{\mathrm{SU}(n)})$. We will find that equality works:

$$M^t = d^\gamma = \gcd(d^\infty, n, M_{\mathrm{SU}(n)})\,. \tag{4.7}$$

Recall from section 1.1 that the fixed points of $P^k_+(\mathrm{su}(n))$ are in one-to-one correspondence with the weights in $P^{k/d}_+(\mathrm{su}(d'))$. In one direction, we project $\phi \in P^k_+(\mathrm{su}(n))$ to $\overline{\phi} \in P^{k/d}_+(\mathrm{su}(d'))$, so that $\phi = (\overline{\phi}, \overline{\phi}, \ldots, \overline{\phi})$ (d times); for the inverse direction, we lift $\nu \in P^{k/d}_+(\mathrm{su}(d'))$ to $\widehat{\nu} = (\nu, \nu, \ldots, \nu) \in P^k_+(\mathrm{su}(n))$. For example, for $\mathrm{SU}(6)/\mathbb{Z}_3$, $\widehat{0} = (\frac{k}{3}; 0, \frac{k}{3}, 0, \frac{k}{3}, 0)$.

Considering (1.17) and (4.5), we find that for fixed ϕ, the values of $q^t_{[\phi,i]}$ for different i are independent, except for the single relation (3.9):

$$\sum_{i=1}^{d} q^t_{[\phi,i]} = 0 \qquad (\mathrm{mod}\ d^\gamma)\,. \tag{4.8}$$

Choose any solution $q^t_{[\widehat{0},i]}$ to (4.8). There are precisely $(d^\gamma)^{d-1}$ of these, which we will show form the subgroup $\mathbb{Z}_{d^\gamma} \oplus \cdots \oplus \mathbb{Z}_{d^\gamma}$ of the charge group K. From Claim (c) we see that, for each $1 \leq \ell < d$,

$$\dim(\Lambda_{\ell d})\, q^t_{[\widehat{0},i]} = q^t_{[\widehat{\Lambda}_\ell,i]}\,. \tag{4.9}$$

Continuing recursively, we obtain all twisted charges $q^t_{[\psi,i]}$. In fact, from Claim (b) we obtain the remarkable formula, valid for all fixed points ϕ:

$$q^t_{[\phi,i]} = \dim(\overline{\phi})\, q^t_{[\widehat{0},i]}\,. \tag{4.10}$$

Now let us determine the untwisted charges. We expect this to be a solution to (1.1) with $M = M_{\mathrm{SU}(n)}$ and $q_{[0]} = 1$. In other words, we expect the total charge group to be

$$K = \mathbb{Z}_{M_{\mathrm{SU}(n)}} \oplus \mathbb{Z}_{d^\gamma} \oplus \cdots \oplus \mathbb{Z}_{d^\gamma} \tag{4.11}$$

($d-1$ copies of $\mathbb{Z}_{d^\gamma}$). Unfortunately at present we can only prove the existence of this solution in special cases (which however exhaust most possibilities for n, k, d). We can however describe a general ansatz for it, and solve it for examples.

We know $q_{[\lambda]} = \dim(\lambda)$ for all non-fixed points $\lambda \in P^k_+$, so it suffices to study the fixed points ϕ. When d and $M_{\mathrm{SU}(n)}$ are coprime, *i.e.* when the largest power d^β dividing $k+n$ obeys $d^\beta < n$, we get from (3.10) the unique general untwisted solution $q_{[\phi,i]} = d^{-1}\dim(\phi)$ where d^{-1} is the inverse of d mod $M_{\mathrm{SU}(n)}$. Thus it suffices to consider d dividing $M_{\mathrm{SU}(n)}$.

Let λ be any weight with $t(\lambda)$ coprime to d. Then the tensor product (hence fusion and NIM-rep (1.17)) of λ with any $J^{d'}$-fixed point ϕ, cannot contain any $J^{d'}$-fixed points, but will expand out into full $J^{d'}$-orbits. This means (1.1) gives

$$\dim(\lambda)\, q_{[\phi,i]} = \frac{1}{d}\dim(\lambda)\dim(\phi) \quad \left(\mathrm{mod}\ \frac{M_{\mathrm{SU}(n)}}{d}\right)\,, \tag{4.12}$$

where the $\frac{1}{d}$ arises because only one orbit representative appears in (1.17). Doing this for all such λ and using Claim (a), we obtain

$$q_{[\phi,i]} = \frac{1}{d}\dim(\phi) + \ell_i(\phi)\,\frac{M_{\mathrm{SU}(n)}}{d^{\gamma+1}}\,, \tag{4.13}$$

where $0 \leq \ell_i(\phi) < d^\gamma$ are integers. Because this untwisted solution must have order dividing $M_{\mathrm{SU}(n)}$, the charges $q_{[\phi,i]}$ must be integers, and thus

$$\frac{M_{\mathrm{SU}(n)}}{d^\gamma}\,\ell_i(\phi) = -\mathrm{dim}(\phi) \qquad (\mathrm{mod}\ d)\,. \tag{4.14}$$

(This is also the reason we took $d^{\gamma+1}$ in (4.13) rather than $d^{\alpha+1}$.) Of course these integers must also obey the condition

$$\sum_{i=1}^{d} \ell_i(\phi) = 0\,. \tag{4.15}$$

These conditions cannot uniquely determine the $\ell_i(\phi)$, as we can always add to them $M_{\mathrm{SU}(n)}/d^\gamma$ times any twisted solution.

We do not have a proof that in general these integers $\ell_i(\phi)$ can be found. However, the untwisted solution *must* look like (4.13), for integers $\ell_i(\phi)$ satisfying (4.14) and (4.15), if that solution is to have order $M_{\mathrm{SU}(n)}$ (the maximal possible). We also know that (3.10) will work mod $M_{\mathrm{SU}(n)}/d$ and that it has $q_{[0]} = 1$. So the only question is whether this solution can be lifted to $M_{\mathrm{SU}(n)}$, modulo twisted solutions.

While we cannot prove this at present, we can give various pieces of evidence in favour of (4.13). First of all, we note that it requires d to divide $\mathrm{dim}(\phi)$ whenever $d^{\alpha+1}$ divides $M_{\mathrm{SU}(n)}$. This is indeed the case as follows from (1.2), (3.2), and Claim (a): if $d^{\alpha+1}$ divides $M_{\mathrm{SU}(n)}$ then for any fixed point ϕ

$$d\,\mathrm{dim}(\phi) = \sum_{\nu} N_{\Lambda_1\,\phi}{}^{\nu}\mathrm{dim}(\nu) = d\sum_{[\nu]} N_{\Lambda_1\,\phi}{}^{\nu} = 0 \qquad (\mathrm{mod}\ d^{\alpha+1})\,. \tag{4.16}$$

Secondly, for the case where $d = n$ we can construct a solution with $M = M_{\mathrm{SU}(n)}$, generalising the solution (2.16) for $\mathrm{SU}(3)/\mathbb{Z}_3$ and proving (4.11) holds for prime n. To this end we put, for any fixed point $\phi \in P_+^k$,

$$q_{[\phi,i]} = \frac{1}{d}\mathrm{dim}(\phi+\Lambda_1) + \ell_i(\phi)\,\frac{M}{d}\,, \tag{4.17}$$

where the integers $0 \leq \ell_i(\phi) < d$ are chosen so that (3.8) is satisfied. It is easy to verify that d does divide $\mathrm{dim}(\phi+\Lambda_1)$. By Claim (b), d will divide each $\mathrm{dim}(\Lambda_i)$ here and so (1.1) will automatically be satisfied. Incidentally, the only significance of the number $\mathrm{dim}(\phi+\Lambda_1)$ in (4.17) or (2.16) is that it is divisible by d and it is congruent to $\mathrm{dim}(\phi)$ mod $M_{\mathrm{SU}(n)}/d$.

Finally, we consider the example $\mathrm{SU}(6)/\mathbb{Z}_3$. It suffices to consider the situation when 3 divides $M = M_{\mathrm{SU}(6)}$, where $M_{\mathrm{SU}(6)} = (k+6)/(2^i 3)$ and $2^i = \gcd(4, k+6)$. The fixed

points are of the form $\phi = (\phi_1, \phi_0, \phi_1, \phi_0, \phi_1)$, where $\phi_0 = k/3 - \phi_1$. Write ϕ_1' for $\phi_1 + 1$ and k' for $k+6$. Then the Weyl dimension formula reads

$$\begin{aligned} \dim(\phi) &= \frac{{\phi_1'}^3 (k'/3 - \phi_1')^2 (k'/3)^4 (k'/3 + \phi_1')^2 (2k'/3 - \phi_1')(2k'/3)^2 (2k'/3 + \phi_1')}{2^4 3^3 4^2 5} \\ &= (k'/3)^6 \frac{{\phi_1'}^3 ((k'/3)^2 - {\phi_1'}^2)^2 ((2k'/3)^2 - {\phi_1'}^2)}{2^6 3^3 5} \, . \end{aligned} \tag{4.18}$$

From this we obtain that $\dim(\phi)$ is always a multiple of $3M$ (in fact $9M$). The ansatz (4.13) then tells us that $q_{[\phi,i]} = \ell_i'(\phi)\, M/3$ for some integers $\ell_i'(\phi)$, and adding an appropriate twisted solution (4.10), we know we can choose $q_{[\phi,i]} = 0$. Indeed, an easy calculation verifies that the sum $\sum_{[\nu]} N_{\Lambda_i \phi}{}^{\nu} \dim(\nu)$ over all J^2-orbits of non-fixed points is divisible by 3 for all fixed points ϕ and fundamentals Λ_i (even though 3 does not divide all $\dim(\nu)$ with $N_{\Lambda_i \phi}{}^{\nu} \neq 0$). Thus the general untwisted solution, for $\mathrm{SU}(6)/\mathbb{Z}_3$ when 3 divides M (*i.e.* $k = 3 \bmod 9$), has $q_{[\phi,i]} = 0$ for each fixed point ϕ. The charge group is therefore indeed $K = \mathbb{Z}_{M_{\mathrm{SU}(6)}} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, as expected from (4.11).

4.3. The case when $d = 2$

When d is even, the theory can be pathological, and in particular, Case B can arise. As usual we compute the NIM-rep coefficients from the ψ matrix. Its only entries that have not yet been given are those between fixed points:

$$\psi_{[\phi,i]\,[\varphi,j]} = \frac{1}{\sqrt{2\,\mathrm{mult}(\varphi)}} \left(S_{\phi\,\varphi} + (-1)^{i+j} e^{-3\pi \mathrm{i} k n/16} \overline{S}_{\overline{\phi}\overline{\varphi}} \right) . \tag{4.19}$$

The strange relative phase $e^{-3\pi \mathrm{i} k n/16}$ is the modular anomaly shift and plays no role in (1.12) or elsewhere in this paper. From (4.19) we readily compute from (1.12) the remaining entries of $\mathcal{N}$:

$$\mathcal{N}_{\Lambda_m\,[\phi,i]}{}^{[\varphi,j]} = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \delta^{ij}\, \overline{N}_{\overline{\Lambda}_{m/2} \overline{\phi}}{}^{\overline{\varphi}} & \text{otherwise.} \end{cases} \tag{4.20}$$

Equations (4.19) and (4.20) hold for both Case A and Case B. In order to show in Case B that (4.20) arises from (4.19) requires (1.10), which implies $S_{\phi\varphi} = 0$ for any order 2 fixed points ϕ, φ. As we have seen before, in Case A the simple current $\mathcal{N}_{J^{d'}0}$ is the identity, but in Case B it satisfies

$$\mathcal{N}_{J^{d'}0\,[\phi,i]}{}^{[\varphi,j]} = \delta^{(2)}_{j,i+1}\, \delta_{\phi,\varphi} \, , \tag{4.21}$$

whenever ϕ, φ are order-two fixed points. Furthermore, $\mathcal{N}_{J^{d'}0\,[\nu]}{}^{[\nu']} = \delta_{[\nu],[\nu']}$, and all other entries are 0.

By (3.4), we find that $\dim(J^{n/2}0) = (-1)^{n/2} \pmod{M_{\mathrm{SU}(n)}}$. When $n/2$ is even, the theory is not pathological, and the analysis and formulae are essentially identical to that in the previous subsection. In particular the matrix $M[d']$ is then a modular invariant for any value of k. We can construct the twisted solution (4.10) as before, and the charge group is therefore very plausibly (4.11) with $M = M_{\mathrm{SU}(n)}$ and $M^t = 2^\gamma$. We can prove that this is the correct answer whenever $M_{\mathrm{SU}(n)}$ is odd since then the arguments of section 3 apply. [The construction of the twisted charges works mod M^t in any case, since the relevant arguments of section 4.2 apply here as well.]

As before, we cannot give a proof in the other cases, but we can at least illustrate our claim with an example. To this end consider $\mathrm{SU}(4)/\mathbb{Z}_2$ for the case when $M_{\mathrm{SU}(4)}$ is even; this can only occur when 4 divides k, in which case $M \equiv M_{\mathrm{SU}(4)} = (k+4)/(2\cdot 3^i)$, where $i = 1, 0$ depending on whether $k+4$ is divisible by 3 or not. The J^2 fixed points are then $\phi^a = (a, k/2-a, a)$ for $0 \leq a \leq k/2$. The twisted solution (4.10) becomes $q^t_{[\phi^a,i]} = (-1)^i(a+1)q^t_{[\phi^0,2]}$ where $i = 1, 2$. This defines a solution mod 2^γ, where $2^\gamma = 4$ if $M/2$ is even, and $2^\gamma = 2$ if $M/2$ is odd. The untwisted solution is given, if $M/2$ is odd, by $q_{[\phi^{2a},i]} = (a+i)M/2$ for $a = 0, 1, \ldots, k/4$ and $i = 1, 2$, and $q_{[\phi^b,j]} = 0$ otherwise. When 4 divides M, $q_{[\phi^a,i]} = M/2$ for $a = 1 \bmod 4$ and $i = 1, 2$, and $q_{[\phi^b,j]} = 0$ otherwise. It is not difficult to check that these charges work, and that this gives rise to the charge group (4.11) with $M^u = M_{\mathrm{SU}(4)}$ and $M^t = 2^\gamma$.

This leaves us with analysing the pathological case, *i.e.* the situation when $n/2$ is odd. As mentioned before, in this case the matrix $M[d']$ will only be a modular invariant if k is even, so let us assume k even in the following. (If k is odd, there are no fixed points, and the analysis of section 3.2 already gives the complete solution.) Furthermore, (3.2) implies that $M = 2$ (if $M_{\mathrm{SU}(n)}$ is even) or $M = 1$ (when it is odd). As the example of SO(3) demonstrates, the charge group will depend on whether we are in Case A (which arises if $k/2$ is even) or in Case B (if $k/2$ is odd).

As we have seen before in section 3.1, in Case A the charge group will be a direct sum of $\mathbb{Z}_2$'s. If $M_{\mathrm{SU}(n)}$ is odd then $\mathbb{Z}_2$ is not possible (the order of any charge group must divide $M_{\mathrm{SU}(n)}$), and the charge group is necessarily trivial. This will happen whenever the exact power of 2 dividing $k+n$, is less than n. The only interesting case is therefore $M_{\mathrm{SU}(n)}$ even, in which case the arguments of section 3.1 imply that $M^u = M^t = 2$. The construction of the twisted solution (4.10) still works, and thus the twisted solution will give one factor of $\mathbb{Z}_2$ to (1.5). We conjecture, as in the previous case, that there always exists an untwisted

solution that works mod 2, *i.e.* that it is non-trivial. Thus we conjecture that the complete charge group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $M_{\mathrm{SU}(n)}$ is even.

In Case B, the twisted solution (4.10) still works mod M^t, where

$$M^t = M = \gcd(2, M_{\mathrm{SU}(n)}) \in \{1, 2\} \,. \tag{4.22}$$

Far less trivial is however the nature of the untwisted solution. For example, the case of SO(3) with k not divisible by 4 shows that the untwisted solution may have order $M^u = 4$, in which case it includes the above twisted solution. On the other hand, the example of $\mathrm{SU}(6)/\mathbb{Z}_2$ (which falls into Case B when $k/2$ is odd) shows that the untwisted solution may not exist altogether: when $k = 2$ mod 16, $M = \gcd(2, M_{\mathrm{SU}(n)}) = 2$, so we would expect an untwisted solution to (1.1) with $q_{[0]} = 1$ and $M = 2$, and hence $q_{[\nu]} = \dim(\nu)$. However, taking $\lambda = \Lambda_1$ and $a = [(0, 0, \frac{k}{2}, 0, 0), i]$, we find that $q_a \in \mathbb{Z} + \frac{1}{2}$ and so the desired untwisted solution would have to have order 4. This contradicts the observation made at the beginning of section 3, that the order M^u must divide $M_{\mathrm{SU}(n)}$. Thus we conclude that the untwisted solution must be trivial, *i.e.* that it has order $M^u = 1$. The total charge group of this theory is then $K = \mathbb{Z}_2$. [Incidentally, this behaviour is special to these values of k: when $k = 6$ mod 8, $M = M^t = 1$ and the charge group is trivial. Also, when $k = 10$ mod 16, the charge group is $K = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, reflecting the separate existence of an untwisted and a twisted solution.]

We conjecture that the total charge group in Case B is (i) trivial if $M_{\mathrm{SU}(n)}$ is odd; is (ii) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $M_{\mathrm{SU}(n)}$ is even and the dimension of $\dim(\widehat{0} + \Lambda_1)$ is even; is (iii) $\mathbb{Z}_4$ if $M_{\mathrm{SU}(n)}$ is divisible by 4 and the dimension of $\dim(\widehat{0} + \Lambda_1)$ is odd; and (iv) is $\mathbb{Z}_2$ if $M_{\mathrm{SU}(n)}$ is even but not divisible by 4 and the dimension of $\dim(\widehat{0} + \Lambda_1)$ is odd. Here $\widehat{0} + \Lambda_1$ is a representative weight of the unique $J^{d'}$-orbit that appears in the fusion of Λ_1 with the fixed point $\widehat{0} = (k/d; 0, \ldots, 0, k/d, 0, \ldots, 0, \ldots, k/d, 0, \ldots, 0)$.

4.4. Some remarks about composite d

We believe that the behaviour of solutions to (1.1) in the special case where d is prime, is quite representative of what happens in general (*i.e.* when d is composite). In particular, simple formulae such as (4.5) continue to hold [20] (though generally with sums over simple currents) for general $\mathrm{SU}(n)/\mathbb{Z}_d$, and so the analysis of the previous subsections can largely follow through.

Equation (4.10) says that all (twisted) fixed point charges can be recovered from those of the basic one $\widehat{0}$, which corresponds to the vacuum in $P_+^{k/d}(\mathrm{su}(d'))$. This continues to hold in all composite examples we have considered, and is the basis for our conjectures in the next section.

There is one new phenomenon which can happen for composite d. Consider for example $\mathrm{SU}(9)/\mathbb{Z}_9$ at level 18, for which $M_{\mathrm{SU}(9)} = 9$. Then (1.1) with $\lambda = \Lambda_3$ and $a = \widehat{0}$ implies

$$84\, q^t_{[\widehat{0},i]} = q^t_{[\widehat{(1;3,2)},i]} \,. \tag{4.23}$$

The fixed point on the left has order 9, while the fixed point on the right has order 3. Thus if we replace i by $i+3$ on the left hand side, we must get the same right hand side. Hence we obtain the constraint that

$$84\,(q^t_{[fp9,i]} - q^t_{[fp9,i+3]}) = 0 \qquad (\mathrm{mod}\ M^t)\,.$$

Since $M^t = 9$ here, this constitutes a nontrivial constraint on the twisted charges. Together with (3.9), this reduces the contribution to K of the twisted solutions from the usual $\mathbb{Z}_9^9$ to $\mathbb{Z}_9^3 \oplus \mathbb{Z}_3^5$. More generally, we get the constraint that M^t must divide $\binom{n}{\ell}(q^t_{[\widehat{0},i]} - q^t_{[\widehat{0},i+\ell]})$ for all ℓ.

5. Conclusions

In this paper we have analysed the charges of D-branes for string theory with target space $\mathrm{SU}(n)/\mathbb{Z}_d$, where $d > 1$ divides n. Based on our results one may conjecture that there are essentially two (or maybe three) different cases that need to be distinguished:

(1) If $(n+1)n/d$ is even (the non-pathological case), then the charge group is a subgroup of

$$K = \mathbb{Z}_{M_{\mathrm{SU}(n)}} \oplus (\mathbb{Z}_{M^t})^{f-1}\ , \tag{5.1}$$

where $f = \gcd(d,k)$ and where

$$M^t = \gcd(d^\infty, n, M_{\mathrm{SU}(n)})\,. \tag{5.2}$$

[Note that if k is coprime to d, the charge group is simply $K = \mathbb{Z}_{M_{\mathrm{SU}(n)}}$. In all examples we have studied, K always contained $\mathbb{Z}_{M_{\mathrm{SU}(n)}}$, but we know of examples — see section 4.4 — where the twisted solutions only gave a proper subgroup of $(\mathbb{Z}_{M^t})^{f-1}$.]

(2) If $(n+1)n/d$ **is odd (the pathological case)**, then the charge group is 'pathologically small', *i.e.* it is bounded as $k \to \infty$. More precisely, when k/f is even ('**Case A**') the charge group is either trivial (if $M_{\mathrm{SU}(n)}$ is odd), or it is a direct sum of at most f copies of the cyclic group $\mathbb{Z}_2$.

When k/f is odd ('**Case B**'), the charge group will be a subgroup of $\mathbb{Z}_M \oplus (\mathbb{Z}_{M^t})^{f-1}$ or $\mathbb{Z}_{MM^t} \oplus (\mathbb{Z}_{M^t})^{f-2}$, where $M = \gcd(2, M_{\mathrm{SU}(n)})$, and f and M^t are given as above.

Our conjectures are the simplest statements we have found which are consistent with all of the examples and results we know. We have not been able to prove these claims in the above generality, but we have been able to prove them for most cases (for example, when d is prime, when k is coprime to d, *etc*). We have also illustrated these results with two examples: $\mathrm{SU}(3)/\mathbb{Z}_3$ is the archetypal example for case (1), while $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ is a good example for case (2). As explained in section 4.4, we expect that a solution to (1.1) is uniquely determined from the vacuum charge $q_{[0]}$, together with the f charges $q_{[\widehat{0},i]}$ associated to the order-f fixed point $\widehat{0} = \sum_{\ell=0}^{f-1} \frac{k}{f} \Lambda_{jn/f}$. These charges are not independent — in particular there is (3.8), and therefore the ambiguity in the construction of the untwisted solution is at most $(\mathbb{Z}_{M^t})^{f-1}$. Also, if d is not prime, there can be additional constraints (see section 4.4).

Some of these observations will generalise to other WZW models. For example, $\mathrm{SO}(2n+1) = \mathrm{B}_n/\mathbb{Z}_2$ will behave like case (2) above for every n and k.

It is intriguing that for the quotient groups of $\mathrm{SU}(n)$, the pathological case (2) appears precisely for those theories where the quantisation condition of [12] is $k \in 2\mathbb{Z}$. This suggests that one should be able to understand the pathological behaviour of the D-brane charges also from a geometrical point of view. At any rate, one should expect that the above results should agree with the charge groups that can be determined by a K-theory analysis, and it would be very interesting to check this. The behaviour of the D-brane charges for non-simply connected groups is far more subtle than in the simply connected case, and this should also be reflected in the K-theory analysis.

It is believed that the K-theory analysis should apply to the supersymmetric version of this model. As is well known, the supersymmetric theory can be rewritten in terms of the bosonic WZW model (at a shifted level) together with $\dim(\bar{\mathfrak{g}})$ free fermions. In this paper we have only analysed the bosonic part in detail; since we have only considered D-branes that preserve the full affine symmetry, it is always possible to choose boundary conditions for the free fermions so that the combined D-brane preserve the full supersymmetry. Thus the D-branes we have discussed here always correspond to supersymmetric branes.

There is very good evidence that the D-brane charges must satisfy (1.1), but it is not so obvious whether this is the only constraint that restricts them. In particular, it is conceivable that there are other constraints that remove some of the ambiguities that are described by the twisted solutions.

Finally, we have only discussed the maximally symmetric D-branes for these theories, but it is clear that there are also 'twisted' D-branes that only preserve the affine symmetry up to an outer automorphism. It should be possible to determine their charges using similar methods. Given the situation for the simply connected case, one should also expect that the full charge lattice will not be generated by these D-branes alone. It would therefore be interesting (as in the simply connected case) to understand the structure of the remaining charges.

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Note added in proof. The K-theory calculation for SO(3) has recently appeared [33]; they recover our charge groups, as expected, but they also find evidence for a second supersymmetric conformal field theory associated to SO(3). A proposed construction of this novel SO(3) model has been given in [34], where it is shown to recover the second family of possible SO(3) charge groups of [33].

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